

Problem Solving with Dimensional Analysis

Feature

by Robert Schmidt and Kevin Housen

A simple but powerful technique allows scientists and engineers to design small-scale experiments to accurately simulate large-scale events, such as nuclear explosions, that are impossible to recreate in the laboratory

Dimensional analysis embodies a simple but subtle concept that provides a powerful, cost-effective approach to solving complex problems in industry. The concept is based on the intuitive notion that an equation must be dimensionally homogeneous, that is, its solution must be invariant to changes in measurement units. Although this concept was apparently recognized as early as 1761 by Francois Daviet de Foncenex, its practical consequences were not fully realized until Edgar Buckingham developed the so-called π -theorem in 1914.

The π -theorem states that if a functional relationship involves a number of variables and N fundamental units of measurement, then the relationship can be rewritten in terms of N fewer arguments that are nondimensional ratios of the original variables. By reducing the number of tests necessary to characterize a phenomenon, the reduction of variables can greatly simplify experimental studies. Analytic studies can also benefit from dimensional analysis, for example, by reducing partial differential equations to ordinary differential equations or by reducing spe-

cific relationships between variables to power laws.

By far the most popular use of dimensional analysis has been in experimental investigations, not only because the technique reduces the number of tests required, but also because it forms the basis for conducting similar experiments. In this context, two experiments are similar if the nondimensional variables identified in the dimensional analysis have the same value in both experiments. Because it is often impractical to conduct experiments under the specific conditions desired, one may wish to redesign the experiment to make it more manageable. For example, testing at reduced size scale can provide a significant cost savings, as is often the case in studies of aerodynamics or fluid mechanics.

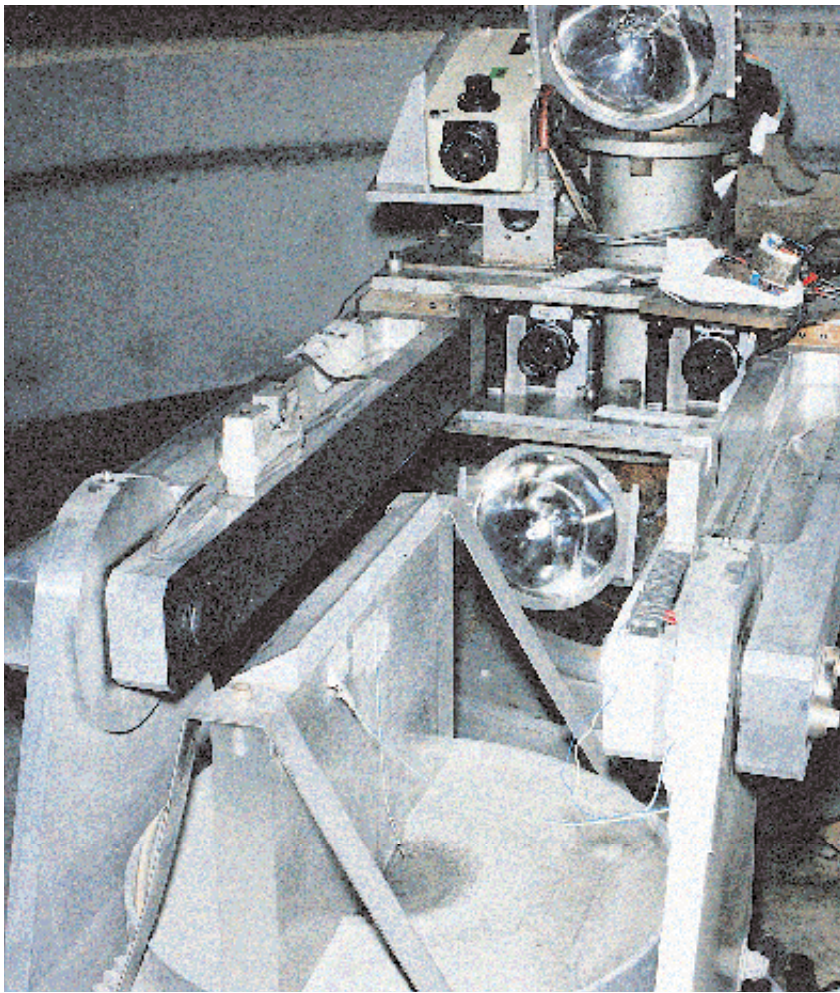
In other instances, such as in the examples given below, it is not only impractical but impossible to perform the desired experiment. In these cases, one can simulate a prototype experiment by designing a model experiment with appropriate test conditions, which may be less expensive or at least attainable. Dimensional analysis identifies the conditions required for similarity and provides the framework within which the results can be applied to the actual problem of interest.

Cratering

Over the past 30 years, the defense community has studied the problem of crater formation by the detonation of conventional and nuclear explosives. At the same time, the formation of hypervelocity impact craters on the surface of Earth and other planets has interested planetary astronomers, especially in light of the recent realization that Earth is subject to periodic catastrophes from the large-scale impacts of asteroids and comets.

Although the physical laws that govern cratering are well-known—the balance of mass, energy and momentum, and the constitutive relations of the materials—the complexity of the process and the wide range of prevailing pressures and temperatures make analytical or even numeri-

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Geotechnical centrifuge, housed at Boeing, allows researchers to simulate in the laboratory the formation of craters by very-high-energy events, such as a nuclear explosion or the impact of a meteoroid. Fig. 1

cal solutions very difficult. Consequently, much of the information on crater formation has come from laboratory and field experiments.

Cratering experiments and the interpretation of their results, however, present their own difficulties. Applications of explosive cratering often involve very energetic events. Field tests can be extremely complex and costly and in some cases are precluded by test ban treaties. Many applications of impact cratering require impactors that are kilometers in size and travel at tens of kilometers per second, whereas laboratory studies are limited to centimeter-size impactors at velocities below about 8 km/s. In cases such as these, where direct experimental tests are either prohibitively expensive or impossible, the only recourse is to conduct experiments under conditions that are attainable

VARIABLE	DESCRIPTION	UNITS
V	crater volume	length^3
m	explosive charge mass	mass
δ	charge density	$\text{mass}/\text{length}^3$
Q	energy/mass charge	$\text{length}^2/\text{time}^2$
ρ	soil density	$\text{mass}/\text{length}^3$
ϕ	soil angle of internal friction	dimensionless
g	gravitational acceleration	$\text{length}/\text{time}^2$

and then to address the question of how to scale the results to the actual problem. This is the type of question for which dimensional analysis proves to be a very powerful tool.

Consider the volume of a crater produced by the detonation of an explosive charge in a cohesionless soil. For simplicity, the burial depth of the charge is assumed to be zero; that is, the charge center is coincident with the ground surface. The seven variables in the table describe this cratering problem.

We could write a functional relationship that involves these variables as

$$V = f(m, \delta, Q, \rho, \phi, g) \quad (1)$$

Because little can be deduced about this functional dependence from physical arguments, it would appear that a large number of tests are required to determine the dependence of V on the other six variables. But dimensional analysis can simplify and reduce the number of separate experiments required. Because the variables include three independent dimensional units—mass, length and time—the π -theorem allows a simpler relationship to be written in terms of just four (seven minus three) nondimensional ratios or “groups.” Although these groups can be formed in any number of ways, the particular choice does not affect the final result. The π -theorem requires only that each variable be included in at least one group and that the groups themselves are each dimensionless. One generally chooses the groups based on some experimental objective, the ease with which groups can be held constant during a set of experiments or how well the results are correlated.

For the case at hand we want to know how the crater volume varies with the mass of the explosive charge. And we would like the equation in a form that allows us to vary the simulated size of the explosive by varying the gravitational acceleration g . These requirements stipulate that one group be a product of charge size and gravity and that other groups be easy to hold constant while this group is varied.

One useful form for the functional relationship among the seven variables is

$$\frac{rV}{m} = f \left[\frac{g}{Q} \left(\frac{m}{r} \right)^{1/3}, \frac{r}{d}, \phi \right] \quad (2)$$

The group on the left side of Equation 2 is a cratering efficiency, the ratio of the excavated soil mass to the charge mass. Note that the π groups do not necessarily have to be recognizable ratios of physical quantities, as we see in the first group on the right side of Equation 2. This group includes the desired product of gravity and charge size in a form that, as shown below, obviates the need for the second group on the right-hand side of the equation, ρ/δ . The soil friction angle, because it is

already a nondimensional quantity, alone forms the third group.

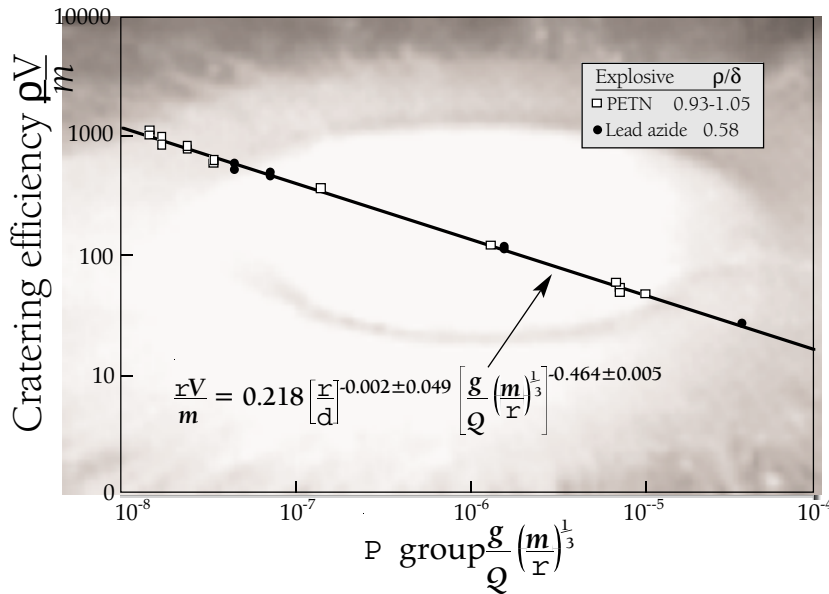
Notice that the effects of the six variables m , δ , Q , ρ , ϕ and g can be determined experimentally by varying only three (nondimensional) parameters, thus providing a considerable reduction in the number of experiments required. Furthermore, one can determine the dependence of crater volume on charge mass by varying $(g/Q)(m/\rho)^{1/3}$, which can be accomplished by varying either m , Q or g .

Because g and m appear as the combination $gm^{1/3}$, a factor of 500 change in g is equivalent to a factor of $500^3 = 1.25 \times 10^8$ variation in m . This equivalence provides a low-cost means of simulating a range of charge mass that would be very expensive to accommodate in field tests. Taking advantage of this outcome, we can effectively vary g by a factor of 500 using a geotechnical centrifuge (Figure 1).

Figure 2 shows centrifuge data from experiments conducted in 1980, along with more recent results. These tests were all performed in dry, cohesionless sand with two types of explosive charges, where ϕ was constant but the density ratio ρ/δ varied by a factor of about two. As shown in the regression equation inset in Figure 2, the cratering efficiency is essentially independent of ρ/δ but varies as a power of $(g/Q)(m/\rho)^{1/3}$. (The observed insensitivity to the ρ/δ group may not be generally applicable to cratering phenomena; for example, this behavior has not been observed in impact cratering.) The power-law dependence and a large body of supporting evidence has led to the important observation that for many cratering-related phenomena the deposition of source energy and momentum occurs as a point source. The point-source concept, along with dimensional analysis, has helped reveal many previously unexplained aspects of cratering.

Orbital debris

We may also apply dimensional analysis to the study of shields used to protect an orbiting spacecraft from meteoroids and man-made debris. These orbital debris shields cause an impacting particle to break apart, melt or vaporize into a dispersed cloud that does not penetrate the spacecraft hull. The problem with simulating the impact of debris on



Cratering efficiency pV/m depends only on the p group $(g/Q)(m/\rho)^{1/3}$, a dimensionless quantity that is a function of the mass of the explosive charge m and gravitational acceleration g . Taking advantage of the π -theorem, one can use the geotechnical centrifuge to vary the gravitational acceleration, thereby effectively varying the charge mass. The data points are from just these kinds of experiments; the inset equation is the linear regression best fit of the data.

the shields is that on-orbit impacts occur at velocities as high as 15 km/s while experimental studies with light gas guns are limited to velocities up to only 8 km/s. The impact velocity in large part determines the phase of the cloud material, and the phase affects the cloud's penetrating power. For example, a cloud of solid fragments can cause significantly more damage than a cloud of vapor. Aluminum debris vaporizes at velocities above about 10 km/s, speeds beyond the capability of conventional particle launchers.

Dimensional analysis has helped to develop a technique for simulating impacts at relatively high velocities. The fundamental idea is to conduct experiments at velocities below 8 km/s, replicating the thermodynamic state and momentum of the cloud produced by the higher-velocity impact under consideration. This is done by replacing the actual impactor and shield materials with surrogates that have correspondingly lower phase-transition specific energies.

Researchers have identified through dimensional analysis the conditions under which a surrogate material experiment correctly replicates the impact of an aluminum particle on an aluminum shield. One can construct a nondimensional group of the form U/\sqrt{h} , where U is the impact velocity and h is the specific heat of melt for the impactor and shield materials. This group allows us to simulate the effects of increasing impact velocity by lowering the specific heat of melt (as long as all other similarity requirements are satisfied).

Cadmium has mechanical and thermodynamic properties that make it an excellent surrogate for aluminum. Because the melt and vaporization phase-transition specific energies are roughly an order of magnitude smaller for cadmium than for aluminum, we can simulate an impact of an alu-

minum particle on an aluminum shield using cadmium counterparts of the same size at a velocity reduced by a factor of 3.1. Thus, cadmium impact tests at velocities up to 8 km/s can simulate aluminum impacts at velocities as high as about 25 km/s. Aluminum impacts at 7 km/s agree quite well with cadmium simulations at 2.3 km/s, demonstrating the utility of the similarity concept. Higher-velocity simulations with cadmium have shown that shield response at high velocities is dramatically different than was predicted earlier from simple extrapolations based on energy or momentum considerations and test data at 7 km/s.

Power laws

Another unappreciated aspect of dimensional analysis is that in some cases it can explain power-law relationships between variables. The recent use of fractals and self-similarity in many fields of science emphasizes that power laws are ubiquitous in nature, and often are not derivable solely from mathematical or physical arguments. However, according to the π -theorem, if there is a relationship among $N+1$ variables involving N independent dimensional units, the relationship can be expressed in terms of a single nondimensional parameter, giving a power-law dependence.

Consider the height h reached by an object lofted vertically with velocity U under the influence of gravitational acceleration g : $h = f(U, g)$. This relationship contains three variables and two independent dimensions (length and time), and can be recast in terms of a single nondimensional group: $hg/U^2 = \text{constant}$. In this simple example, dimensional analysis gives the power law relationship between h , U and g . Although the power-law exponents derived in this manner are often rational numbers, this is not always the case, as illustrated by the point-source solutions to the impact and explosion cratering problem.

Word to the wise

Dimensional analysis is not without drawbacks. Omission of an important variable may lead to erroneous conclusions. For example, early uses of dimensional analysis in cratering omitted the specific energy of the source, which led to disparities between crater scaling theories and experiments that persisted for more than two decades. The analysis for lofting an object as discussed above, would certainly be incorrect if air drag were important.

Omission of one or more variables does not necessarily invalidate a result, however, but may instead merely restrict its generality. For example, consider an analysis that includes two material properties, one with units of density and the other units of stress. Any additional density or stress measures (or any variable with units, such as velocity, that can be derived from density and stress)

could be omitted as long as one restricts the application to a given type of material. This is true because the additional properties would appear as dimensionless ratios that remain constant as long as the material type is also held constant.

In so-called model-of-model testing—a valuable technique developed from dimensional analysis that can be used to test a relationship—one changes the values of the individual variables that make up a nondimensional group while holding the value of the group and all other groups constant. If the functional relationship is correct, the tests will give the same answer. In the cratering example discussed above, one can perform two tests having a constant value of $(g/Q)(m/\rho)^{1/3}$, one with given values of m and g , and the other with values of $m/8$ and $2g$. If the relationship is complete, the two tests will produce the same cratering efficiency. On the other hand, if an important variable with units of, say, length were neglected (for instance the mean grain size of the soil tested), the variation in charge size would expose the neglected dependence because the ratio of grain size to charge size would be different in the two tests.

Dimensional analysis is a useful but often neglected tool that can simplify both the analysis and application of experimental results. In its simplest use, dimensional analysis can greatly reduce the number of tests needed to characterize a phenomenon. An additional benefit is the identification of similarity requirements, which helps determine the dependence on a variable that is expensive to change by substituting a “less expensive” variable. While it is used commonly in some fields, such as fluid mechanics and civil engineering, the use of dimensional analysis as a general problem-solving tool in physics has been more limited. As Buckingham stated 30 years ago, “It is hoped that the few sample illustrations of its use which have been given will prove interesting to physicists who have not been in the habit of making much use of dimensional reasoning.”^v

RECOMMENDED READING

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